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Test functions space in noncommutative quantum field theory

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Abstract: It is proven that the $\star$-product of field operators implies that the space of test functions in the Wightman approach to noncommutative quantum field theory is one of the Gel'fand-Shilov spaces S^{β} with $\beta < 1/2$. This class of test functions smears the noncommutative Wightman functions, which are in this case generalized distributions, sometimes called hyperfunctions. The existence and determination of the class of the test function spaces in NC QFT is important for any rigorous treatment in the Wightman approach.

Keywords: Non-Commutative Geometry, Space-Time Symmetries.

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Contents

1. Introduction

Quantum field theory (QFT) as a mathematically consistent theory was formulated in the framework of the axiomatic approach in the works of Wightman, Jost, Bogoliubov, Haag and others ([1]–[4]). Noncommutative quantum field theory (NC QFT), as one of the generalizations of standard QFT, has been intensively developed during the recent years (for a review, see [5]). The idea of such a generalization of QFT ascends to Heisenberg and it was first put forward in [6]. The present development in this direction is connected with the construction of noncommutative geometry [7] and new physical arguments in favour of such a generalization of QFT [8]. Essential interest in NC QFT is also connected with the fact that in some cases it is obtained as a low-energy limit from the string theory [9]. The simplest and at the same time most studied version of noncommutative theory is based on the Heisenberg-like commutation relations between coordinates,

$$[\hat{x}_\mu, \hat{x}_\nu] = i\,\theta_{\mu\nu}, \tag{1.1}$$

where $\theta_{\mu\nu}$ is a constant antisymmetric matrix.

NC QFT can be formulated also in commutative space by replacing the usual product of operators by the star (Moyal-type) product:

$$\varphi(x) \star \varphi(x) = \exp\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x^\mu}\,\frac{\partial}{\partial y^\nu}\right)\varphi(x)\varphi(y)|_{x=y}. \tag{1.2}$$

This product of operators can be extended to the corresponding product of operators in different points as well as for an arbitrary number of operators:

$$\varphi(x_1) \star \cdots \star \varphi(x_n) = \prod_{a<b} \exp\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x_a^\mu}\,\frac{\partial}{\partial x_b^\nu}\right)\varphi(x_1)\ldots\varphi(x_n); \qquad a,b = 1,2,\ldots,n. \tag{1.3}$$

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Let us stress that actually the field operator given at a point is not a well-defined operator (see [1]–[3]). Well-defined operators are the smoothed operators:

$$\varphi_f \equiv \int \varphi(x)\, f(x)\, d\,x, \tag{1.4}$$

where $f(x)$ is a test function. In QFT the standard assumption is that $f(x)$ are test functions of tempered distributions. Nevertheless in a series of papers (see [10]–[12] and references therein), the axiomatic approach to QFT was developed for other spaces of test functions.

Wightman approach in NC QFT was formulated in [13, 14] (see also [15]). For a theory described by the Hermitian field $\varphi(x)$ and with the vacuum state denoted by Ψ_0, the Wightman functions can be formally written down as follows:

$$W_\star(x_1, x_2, \ldots, x_n) = \langle \Psi_0, \varphi(x_1) \star \cdots \star \varphi(x_n) \Psi_0 \rangle. \tag{1.5}$$

The formal expression (1.5) actually means that the scalar product of the vectors $\Phi = \varphi_{f_k} \star \cdots \star \varphi_{f_1} \Psi_0$ and $\Psi = \varphi_{f_{k+1}} \star \cdots \star \varphi_{f_n} \Psi_0$ is the following:

$$\begin{aligned}\langle\, \Phi, \Psi \,\rangle &= \langle\, \Psi_0, \varphi_{f_1} \star \cdots \star \varphi_{f_n} \Psi_0 \,\rangle \\ &:= \int \langle\, \Psi_0, \varphi(x_1) \star \cdots \star \varphi(x_n)\, \Psi_0 \,\rangle f_1(x_1) \cdots f_n(x_n)\, dx_1 \cdots dx_n \\ &= \int W_\star(x_1, x_2, \ldots, x_n)\, f_1(x_1) \cdots f_n(x_n)\, dx_1 \cdots dx_n \\ &= \int W(x_1, \ldots, x_n)\, f_1(x_1) \star \cdots \star f_n(x_n)\, d\,x_1 \ldots d\,x_n,\end{aligned} \tag{1.6}$$

where $W(x_1, \ldots, x_n) = \langle \Psi_0, \varphi(x_1) \cdots \varphi(x_n) \Psi_0 \rangle$ and the last equality is achieved by repeatedly using the definition of the derivative of a distribution,

$$\int \varphi'(x) f(x)\, dx = -\int \varphi(x) f'(x)\, dx$$

and by the convergence of the star-product series. This choice of the product of operators φ_{f_1} and φ_{f_2} is compatible with the twisted Poincaré invariance of the theory [16, 17] and also reflects the natural physical assumption, that noncommutativity should change the product of operators not only in coinciding points, but also in different ones. This follows also from another interpretation of the Heisenberg-like commutation relations in NC QFT in terms of a quantum shift operator [18]. Remark that although the expression $W(x_1, \ldots, x_n) = \langle \Psi_0, \varphi(x_1) \cdots \varphi(x_n) \Psi_0 \rangle$ looks exactly like in the commutative case, the fields $\varphi(x_i)$ are Heisenberg fields, in this case noncommutative ones, and they carry all the characteristics of noncommutative interaction, e.g. nonlocality in the noncommutative directions and broken Lorentz invariance. The noncommutative Wightman functions are, however, twisted Poincaré scalars (while the noncommutative Heisenberg fields are twisted Poincaré covariants).

The aim of this paper is to find the class of test functions, for which the $\star$-product (1.2), with the extension (1.3), is well defined. We should point out that, besides the definition (1.2) of the Moyal $\star$-product, another form exists, the so-called "integral representation". The two forms are not entirely equivalent. The integral representation, or "nonperturbative" definition of the Moyal product is the correct one in the Moyal treatment of Quantum Mechanics. Based on it, a rigurous study of the algebras of distributions in Quantum Mechanics was done in [19] and rigurous relations between the integral representation and the "asymptotic expansion" (formally given by (1.2)) of the Moyal $\star$-product were established in [20]. In NC QFT, it is the asymptotic expansion of the $\star$-product which is fundamental, especially if we think in terms of the twisted Poincaré symmetry, which leads naturally to the form (1.2), with the extension (1.3), of the $\star$-product (see [16]). Since we are interested in an axiomatic formulation of NC QFT with twisted Poincaré symmetry, our purpose in this paper is to study under which general conditions (i.e. for which space of test functions) the asymptotic series implied by (1.2) converges. It will turn out that the space of test functions which insures the convergence of the asymptotic expansion is more restrictive than the one found for the nonperturbative case (integral representation). In what follows, by $\star$-product we shall understand eq. (1.2) with the extension (1.3).

We shall prove that in order for the $\star$-multiplication to be well defined for the functions $f_i\,(x_i)$, it is necessary and sufficient that

$$f_i\,(x_i) \in S^{\beta}, \qquad \beta < 1/2. \tag{1.7}$$

S^{β} is a Gel'fand-Shilov space [21]. The case $\beta = 1/2$ is not excluded, but the corresponding constant B (see ineq. (2.7)) has to be sufficiently small. We also show that after the $\star$-multiplication we obtain functions which belong to the same space S^{β} with the same β as $f_i\,(x_i)$. In other words, we prove that eq. (1.6) implies that

$$\langle\, \Phi, \Psi \,\rangle = \int W\,(x_1, \ldots, x_n)\, f_\star\,(x_1, \ldots, x_n)\, d\,x_1 \ldots d\,x_n, \tag{1.8}$$

where $f_\star\,(x_1, \ldots, x_n) \equiv f_1\,(x_1) \star f_2\,(x_2) \star \ldots \star f_n\,(x_n)\, \in S^{\beta},\ \beta < 1/2$.

First we consider the case of space-space noncommutativity ($\theta^{\,0i} = 0$). This case is free from the problems with causality and unitarity [22]–[27] and in this case the main axiomatic results: CPT and spin-statistics theorems, Haag's theorem remain valid [13]–[15, 17, 27]–[28]. Then we show that all calculations can be easily extended to the general case $\theta^{\,0i} \neq 0$ and moreover the obtained results remain true in the general case as well.

These results are crucial for the derivation of the reconstruction theorem in NC QFT. This problem will be considered in a future work.

2. Basic Statements

Let us point out that (as in the commutative case) the operator φ_f acts on the space J, which is a span of all sequences of the type:

$$g = \{f_0, f_1\,(x_1), f_2\,(x_1, x_2), \ldots, f_n\,(x_1, \ldots x_n)\}, \tag{2.1}$$

where $f_0 \in \mathbb{C}$, $f_k\,(x_1, \ldots x_i, \ldots x_k)$ is a function of k variables, $x_i \in \mathbb{R}^4$. The sum of vectors and their multiplication by complex numbers are defined in the standard way [1]–[3].

By definition, the operator φ_f is determined as follows

$$\varphi_f \{f_0, f_1, f_2 \dots f_n\} = \{f f_0, f \star f_1, f \star f_2, \dots, f \star f_n\}, \tag{2.2}$$

$$f \equiv f(x), \quad f_k \equiv f_k(x_1, \dots x_k), \quad f \star f_k \equiv f \, \exp\left(\frac{i}{2} \theta^{\mu\nu} \frac{\overleftarrow{\partial}}{\partial x^\mu} \frac{\overrightarrow{\partial}}{\partial x_1^\nu}\right) f_k. \tag{2.3}$$

The special set of vectors

$$\{\Psi_0, \, \varphi_{f_1} \Psi_0, \, \dots, \varphi_{f_1} \star \dots \star \, \varphi_{f_n} \Psi_0\}, \tag{2.4}$$

where $\Psi_0 \equiv \{1, 0, \dots 0\}$, $f_i \equiv f_i(x_i)$, $x_i \in \mathbb{R}^4$, forms a dense domain J_0 in the space under consideration. The scalar product in J_0 is defined by Wightman functions (see eq. (1.6)).

The necessary condition

$$\langle \Phi, \Psi \rangle = \overline{\langle \Psi, \Phi \rangle} \tag{2.5}$$

is satisfied (just as in the commutative case) if

$$W_\star(x_1, \cdots, x_n) = \overline{W_\star(x_n, \cdots, x_1)}. \tag{2.6}$$

In fact, due to the antisymmetry of $\theta^{\mu\nu}$, condition (2.6) leads to (2.5).

Let us recall that the scalar product between any two vectors in J is defined by a finite linear combination of Wightman functions with an arbitrary accuracy. This fact is crucial in the derivation of axiomatic results both in commutative and noncommutative cases.

Our aim is to determine the spaces on which eq. (1.6), that is the $\star$-multiplication, is well-defined. Evidently the space of tempered distributions cannot be compatible with the $\star$-multiplication, as each function of this space admits only a finite number of derivatives [1]. Gel'fand and Shilov proved that if $f(x) \in S^\beta$ (see ineq. (2.7)) then the series of derivatives of infinite order is well-defined on such a space. Thus we assume that $f(x) \in S^\beta$ and prove that the $\star$-product is well-defined only if each f_i belongs to the Gel'fand-Shilov space S^β, $\beta < 1/2$. The $\star$-product is well-defined also if $\beta = 1/2$, but only for functions which satisfy inequality (2.7) with sufficiently small B.

Let us recall the definition and basic properties of Gel'fand-Shilov spaces S^β [21]. In the case of one variable $f(x)$, $x \in \mathbb{R}^1$ belongs to the space S^β, if the following condition is satisfied:

$$\left| x^k \frac{\partial^q f(x)}{\partial x^q} \right| \leq C_k B^q q^{q\beta}, \quad -\infty < x < \infty, \quad k, q \in \mathbb{N}, \tag{2.7}$$

where the constants C_k and B depend on the function $f(x)$. Below we use the inequality (2.7) only at $k = 0$:

$$\left| \frac{\partial^q f(x)}{\partial x^q} \right| \leq C B^q q^{q\beta}, \quad -\infty < x < \infty, \quad q \in \mathbb{N}. \tag{2.8}$$

In the case of a function of several variables, the inequality (2.8) holds for any partial derivative:

$$\left| \frac{\partial^q f(x^1, \dots x^k)}{(\partial x^i)^q} \right| \leq C B^q q^{q\beta}, \quad -\infty < x_i < \infty, \quad q \in \mathbb{N}. \tag{2.9}$$

As our results do not depend on the constant C, in what follows we put $C = 1$.

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3. Proof of the Statement

We point out that if the $\star$-product is well-defined for $f_i\,(x_i)\,\star\,f_{i+1}\,(x_{i+1})$, it is also well-defined for all expressions in the right-hand side of eq. (1.6).

Let us stress that, in fact, under the same conditions

$$\exp\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x^{\mu}}\,\frac{\partial}{\partial y^{\nu}}\right)\,f\,(x,y)$$

is well-defined as well, if $f\,(x,y)\in S^{\beta}$, $\beta<1/2$.

First we consider the case when the $\star$-multiplication acts only on the coordinates x_i^1 and x_i^2, i.e. the case of space-space noncommutativity with only $\theta^{12}=-\theta^{21}\neq 0$. Then we show that the proof can be extended to the general case $\theta^{\,0i}\neq 0$ and the corresponding results are the same.

Let us study

$$f\,(x)\,\star\,f\,(y)=\exp\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x^{\mu}}\,\frac{\partial}{\partial y^{\nu}}\right)\,f\,(x)\,f\,(y). \tag{3.1}$$

We have to find the conditions under which the series

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x^{\mu}}\,\frac{\partial}{\partial y^{\nu}}\right)^{n}\,f\,(x)\,f\,(y)\equiv\sum_{n=0}^{\infty}\frac{D_n}{n!} \tag{3.2}$$

converges. Evidently

$$D_n\;=\;\sum_{n_\mu,n_\nu}\left(\frac{i}{2}\theta^{\,\mu\nu}\right)^{n_\mu}\left(\frac{i}{2}\theta^{\,\nu\mu}\right)^{n_\nu}\frac{\partial^{\,n_\mu}}{(\partial x^{\mu})^{n_\mu}}\frac{\partial^{\,n_\mu}}{(\partial y^{\nu})^{n_\mu}}\frac{\partial^{\,n_\nu}}{(\partial x^{\nu})^{n_\nu}}\frac{\partial^{\,n_\nu}}{(\partial y^{\mu})^{n_\nu}}\,f\,(x)\,f\,(y), \tag{3.3}$$

$$\mu,\nu=1,2,\quad n_\mu+n_\nu=n.$$

We show that actually it is sufficient to estimate one term in the series (3.3), that is to estimate the module of

$$\theta^{\,n}\,\frac{\partial^{\,n_\mu}}{(\partial x^{\mu})^{n_\mu}}\,\frac{\partial^{\,n_\mu}}{(\partial y^{\nu})^{n_\mu}}\,\frac{\partial^{\,n_\nu}}{(\partial x^{\nu})^{n_\nu}}\,\frac{\partial^{\,n_\nu}}{(\partial y^{\mu})^{n_\nu}}\,\,f\,(x)\,f\,(y)\equiv\theta^{\,n}\,B\,(n_\mu,n_\nu),\quad\theta\equiv\left|\frac{i\,\theta^{\,\mu\nu}}{2}\right|. \tag{3.4}$$

Using the inequality (2.9) and taking into account that $n_\mu,n_\nu\leq n$, we obtain

$$|B\,(n_\mu,n_\nu)|<B^{\,2\,(n_\mu+n_\nu)}\,{n_\mu}^{2\beta n_\mu}\,{n_\nu}^{2\beta n_\nu}<n^{2\beta\,(n_\mu+n_\nu)}B^{\,2n}=n^{2\beta\,n}\,B^{\,2n}, \tag{3.5}$$

as $n_\mu+n_\nu=n$. This estimate is one and the same for any n_μ,n_ν. As the total number of these terms in D_n is 2^n, taking into account that the module of the sum is less than the sum of the moduli, we come to the following inequality

$$|D_n|<\left(2\,\theta\,B^2\right)^n\,n^{2\,n\,\beta}. \tag{3.6}$$

Using this inequality and the fact that, according to the Stirling formula, $\frac{1}{n!}<\left(\frac{e}{n}\right)^n$, we come to the estimate

$$\left|\frac{D_n}{n!}\right|<\tilde{B}^n\,n^{-2\,n\,\gamma}, \tag{3.7}$$

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where $\tilde{B} = 2\, e\, \theta\, B^2,\ \gamma = 1/2 - \beta$.
For any $\tilde{B}$ the series

$$\sum_{n=0}^{\infty} \tilde{B}^n\, n^{-2\, n\, \gamma} \tag{3.8}$$

converges if $\gamma > 0$, i.e. $\beta < 1/2$, and diverges if $\beta > 1/2$. If $\beta = 1/2$ the series converges if $\tilde{B} < 1$.

Thus we come to the conclusion that the series (3.2) for arbitrary B and C is convergent if $\beta < 1/2$ and divergent if $\beta > 1/2$. If $\beta = 1/2$ the series converges at sufficiently small B.

Now let us show that the function $f_\star\,(x, y) \equiv f\,(x) \star f\,(y)$ belongs to the same Gelfand-Shilov space S^β, $\beta < 1/2$ as $f\,(x)$.

According to the inequality (2.9) we have to prove that

$$\left| \frac{\partial^{\,q}}{(\partial x^\mu)^q}\, f_\star\,(x, y) \right| < C'\, B'^q\, q^{q\,\beta}, \quad -\infty < x^\mu < \infty \tag{3.9}$$

with some constants C' and B'.

As above, first we consider the corresponding estimate for one term $B\,(n_\mu, n_\nu)$ in D_n (see eqs. (3.3), (3.4)). Actually we have to estimate

$$\frac{\partial^{\,n_\mu + q}}{(\partial x^\mu)^{n_\mu + q}}\, f\,(x).$$

In accordance with eq. (2.9)

$$\left| \frac{\partial^{\,n_\mu + q}}{(\partial x^\mu)^{n_\mu + q}}\, f\,(x) \right| < B^{n_\mu + q}\, (n_\mu + q)^{(n_\mu + q)\beta}. \tag{3.10}$$

Evidently,

$$(n_\mu + q)^{(n_\mu + q)\beta} = q^{q\beta}\, {n_\mu}^{n_\mu \beta} \left(1 + \frac{1}{x}\right)^{x q \beta} (1 + x)^{\frac{1}{x} n_\mu \beta}, \quad x = \frac{n_\mu}{q}.$$

After elementary calculations we obtain

$$(n_\mu + q)^{n_\mu + q} < q^{q\beta}\, {n_\mu}^{n_\mu \beta}\, e^{q\beta}\, e^{n_\mu \beta} < e^{q\beta}\, q^{q\beta}\, e^{n\beta}\, n^{n_\mu \beta}. \tag{3.11}$$

Thus according to the ineq. (3.10) we have

$$\left| \frac{\partial^{\,n_\mu + q}}{(\partial x^\mu)^{n_\mu + q}}\, f\,(x) \right| < \left(B e^\beta\right)^q q^{q\beta} B^n e^{n\beta} n^{n_\mu \beta}. \tag{3.12}$$

Comparing the bounds (3.7) and (3.12), we see that

$$\left| \frac{\partial^{\,q}}{(\partial x^\mu)^q}\, B\,(n_\mu, n_\nu) \right| < \left(B e^\beta\right)^q q^{q\beta} B^{2n} e^{n\beta} n^{n\beta}. \tag{3.13}$$

Thus

$$\frac{\partial^{\,q}}{(\partial x^\mu)^q}\; f_\star\,(x, y) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^{\,q} D_n}{(\partial x^\mu)^q} \tag{3.14}$$

admits the following estimate

$$\left|\frac{\partial^q}{(\partial x^\mu)^q}\, f_\star(x,y)\right| < (Be^\beta)^q\, q^{q\beta} \sum_{n=0}^{\infty} \frac{1}{n!}\,(B^2 e^\beta)^n\, n^{n\beta}. \tag{3.15}$$

The series in the right-hand side of eq. (3.15) converges if $\beta < 1/2$ and diverges if $\beta > 1/2$ on the same grounds as the series (3.8). Thus we see that the necessary condition on $f_\star(x,y)$ (3.10) is fulfilled with

$$B' = B\, e^\beta \quad \text{and} \quad C' = \sum_{n=0}^{\infty} \frac{1}{n!}\,(B^2 e^\beta)^n\, n^{n\beta}.$$

Let us proceed to the general case $\theta^{\,0i} \neq 0$. We prove that in this case the $\star$-multiplication is well defined (just as in the case of space-space noncommutativity) if the corresponding test functions belong to S^β, $\beta < 1/2$.

In order to show this, let us rewrite the condition (2.9) in a more general form, using the standard notations [1, 21]. By definition, $f(x) \in S^\beta, x \in \mathbb{R}^k$ if

$$\left|\frac{D^q}{\partial x^q}\right| < C\, B^q\, q^{q\beta}, \tag{3.16}$$

where

$$x^q = (x^1)^{q_1} \ldots (x^k)^{q_k} \quad D^q = \frac{\partial^{\,|q|}}{(\partial x^1)^{q_1} \ldots (\partial x^k)^{q_k}}, \quad |q| = q_1 + q_2 + \ldots + q_k.$$

As before we consider one term in the expansion (3.3) (now μ and ν possess the values $0, 1, 2, 3$).

Since condition (3.16) contains only q, we can use our previous consideration for an arbitrary term in D_n and come to a similar estimate (but now $\theta = \max|\theta^{\,\mu\nu}/2|$). This estimate leads to the corresponding one on D_n, the only difference being that it is necessary to substitute 2^n by 4^n since $\theta^{\,\mu\nu}$ by rotation can always be reduced to a block-diagonal form with four nonzero components (e.g. $\theta_{01} = -\theta_{10} = \theta$ and $\theta_{23} = -\theta_{32} = \theta'$, all the other components being zero). As the set (3.8) converges at arbitrary $\tilde{B}$, if $\beta < 1/2$, and diverges if $\beta > 1/2$, the results obtained for space-space noncommutativity remain true also in the general case. If $\beta = 1/2$ the words "sufficiently small B" have different meanings in the cases $\theta^{\,0i} = 0$ and $\theta^{\,0i} \neq 0$.

4. Conclusions

In this paper we have proven that the space of test functions for which the Moyal $\star$-product is well-defined, in other words, the space of test functions for the Wightman distribution functions corresponding to the NC QFT, is one of the Gel'fand-Shilov spaces S^β with $\beta < 1/2$. This class of test functions smears the NC Wightman functions which are generalized distribution functions, called sometimes hyperfunctions.

The existence and determination of the class of test functions spaces is important for any rigorous treatment of the axiomatic approach to NC QFT via NC Wightman

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functions and the derivation of rigorous results such as CPT and spin-statistics theorems, and is also needed for the derivation of other results in axiomatic approach such as the cluster-decomposition property of NC Wightman functions and eventually the proof of the reconstruction theorem.

Recall that the class of test functions in the ordinary QFT contains functions with compact support. In the case of NC Wightman functions, however, the set of test functions consists of functions with non-compact support only in the NC coordinates.

Note, however, that in the case of space-space noncommutativity, i.e. $\theta^{0i} = 0$, the test functions can still have finite support in the commutative directions x_0, x_3. As a result, the local commutativity condition can be formulated in these directions as

$$\varphi_{f_1} \star \varphi_{f_2} = \varphi_{f_2} \star \varphi_{f_1}, \tag{4.1}$$

where the test functions $f_1(x)$ and $f_2(x')$ are zero everywhere except on space-like separated finite domains O and O' in the commutative coordinates, i.e. for each pair of points $x \in O$ and $x' \in O'$

$$\left(x_0 - x_0{}'\right)^2 - \left(x_3 - x_3{}'\right)^2 < 0, \tag{4.2}$$

but without any restriction in the noncommutative directions x_1 and x_2. This is in effect the well-known light-wedge locality condition [13, 14].

It should be noted that, although according to the twisted Poincaré symmetry of NC QFT, the Wightman functions should be defined with $\star$-product as in (1.5), for practical purposes one may as well define them with usual product, because the nonlocality is taken into account by the very definition of the Heisenberg fields [13]. Therefore, although in the smeared noncommutative Wightman functions the test functions will not be multiplied with $\star$-product, they still have to belong to the Gel'fand-Shilov space found in this paper, to account for the character of generalized distributions of the noncommutative Wightman functions[1].

In the general case $\theta^{0i} \neq 0$, we have also shown that the space of test functions is the same Gel'fand-Shilov space S^{β} with $\beta < 1/2$, but with respect to all noncommutative coordinates. Thus in the general case we have not the standard condition of local commutativity. As CPT and spin-statistics theorems have been proven for very general spaces [10] we can conclude that the finding of the class of the spaces, in which $\star$-multiplication and thus noncommutative Wightman functions are well-defined, gives rise to the possibility to prove CPT-theorem and, maybe, also spin-statistics theorem for the general case. It should be mentioned that the twisted Poincaré symmetry provides an answer at least for the spin-statistics relation in the case of general noncommutativity: the spin-statistics relation holds for NC QFT with time-space noncommutativity, provided that such theories

[1] In a recent paper [29] it was argued that due to the translational invariance of NC QFT, the commutative and noncommutative Wightman functions are practically the same, with similar properties, leading to the fact that commutative and noncommutative QFT are actually identical. We argue here that this cannot be the case, since the spaces of test functions in the two situations are completely different, emphasizing the deep qualitative difference between the commutative Wightman functions (tempered distributions) and the noncommutative Wightman functions (generalized distributions, i.e. hyperfunctions).

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can be consistently defined [30]. Previous perturbative studies have shown however that such theories are pathological [22]–[27]. The axiomatic study of time-space noncommutative theories based, among other aspects, on the space of test functions found in this work, may resolve the problem, indicating whether or not the time-space noncommutative theories are well defined and the pathologies are mere artifacts of the perturbation theory.

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